stichting
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AFDELING NUMERIEKE WISKUNDE
(DEPARTMENT OF NUMERICAL MATHEMATICS)

NW 59/78 AUGUSTUS

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ON THE STABILITY OF MULTISTEP FORMULAS FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 65R05

On the stability of multistep formulas for Volterra integral equations of the second kind*)

by

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ABSTRACT

The purpose of this paper is to analyse the stability properties of a class of multistep methods for second kind Volterra integral equations. Our approach follows the usual analysis in which the kernel function is a priori restricted to a special class of test functions. In most stability investigations these test functions only depend on the unknown function. In this paper we only require that the kernel function can be separated with respect to its first argument. Stability conditions will be derived. It turns out that in the case of linear convolution kernels the trapezoidal rule for example may become unstable.

KEY WORDS & PHRASES: *Numerical analysis, Volterra integral equations of the second kind, stability.*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Suppose we are given the system of non-linear Volterra integral equations

$$(1.1) \qquad f(x) = g(x) + \int_{x_0}^{x} K(x,y,f(y))dy, \qquad x_0 \le x \le X$$

where g and K are given vector functions and f is the unknown vector function.

Several numerical methods have been proposed to solve this equation, the most familiar one of which is based on a direct quadrature rule. These methods have the form

$$(1.2) \qquad f_{n+1} = g(x_{n+1}) + \sum_{j=0}^{n+1} w_{n+1,j} K(x_{n+1},x_j,f_j) \ , \qquad n \ge k-1,$$

where $f_0,f_1,\ldots$ are approximations to $f(x_0),f(x_1),\ldots$ and $w_{n,j}$ are given weight parameters. We assume that the vectors $f_0,\ldots,f_{k-1}$ have been computed by some adequate starting procedure.

The stability analysis of this and other schemes is carried out either by restricting the kernel function K to some special class of functions - generally the function K = af where a is a complex constant (cf. MAYERS [5], BAKER and KEECH [1]) - or by deriving conditions for small values of the integration steps $h_n = x_{n+1} - x_n$ (cf. KOBAYASI [4] and NOBLE [6]). In this paper, the first approach will be followed. The kernel functions K(x,y,f) will be required to belong to the class of functions of which the variable x can be separated from the variables y and f. Generalizing the analysis presented in [3], the variational equation of scheme (1.2) can be converted into a system of fixed-term recurrence relations by extending the space of perturbations of f_j with additional perturbations. In addition, by restricting K to functions which are linear in x, a fixed-term recurrence relation will be derived only in terms of perturbations of f_j.

2. DERIVATION OF RECURRENCE RELATIONS

For sufficiently small perturbations Δf_j, the variational equation of (1.2) is of the form

$$\Delta f_{n+1} = \sum_{j=0}^{n+1} w_{n+1,j} \frac{\partial K}{\partial f} (x_{n+1}, x_j, f_j) \Delta f_j, \qquad n \geq k-1. \tag{2.1}$$

In order to obtain a fixed-term recurrence relation for the perturbations Δf_j we impose two conditions. Firstly, the arguments x, y and f in the Jacobian matrix are assumed to be separable according to the formula

$$\frac{\partial K}{\partial f} (x,y,f) = P(y,f) + \sum_{i=1}^{r} Q_i(x) R_i(y,f), \tag{2.2}$$

where P and R_i are arbitrary matrices only depending on y and f, and Q_i is an arbitrary matrix only depending on x. Secondly, the weights $w_{n,j}$ are assumed to satisfy the relation

$$\begin{aligned} &\sum_{\ell=0}^{k} a_\ell w_{n+1-\ell,j} = 0 \qquad j = 0,1,\ldots,n-k;\ n \geq 2k-1. \\ &\sum_{\ell=0}^{k} a_\ell = 0 \end{aligned} \tag{2.3}$$

where the parameters a_ℓ are independent of j and k is a positive integer.

From condition (2.2) it follows that

$$\Delta f_{n+1-\ell} = \sum_{j=0}^{n+1} w_{n+1-\ell,j} [P(x_j, f_j) + \sum_{i=1}^{r} Q_i(x_{n+1-\ell}) R_i(x_j, f_j)] \Delta f_j \tag{2.4}$$

$$n+1-\ell \geq k;\ \ell = 0,1,\ldots,k,$$

where $w_{n,j} = 0$ for $j > n$. By virtue of (2.3) Δf_n satisfies the relation

$$\begin{aligned} &\sum_{\ell=0}^{k} [a_\ell I + b_\ell h_n P(x_{n+1-\ell}, f_{n+1-\ell})] \Delta f_{n+1-\ell} = \\ &= \sum_{i=1}^{r} \sum_{\ell=0}^{k} \sum_{j=0}^{n+1} a_\ell w_{n+1-\ell,j} Q_i(x_{n+1-\ell}) R_i(x_j, f_j) \Delta f_j \end{aligned} \tag{2.5}$$

where we have written

$$(2.6) \qquad \sum_{\ell=0}^{k} a_\ell w_{n+1-\ell,j} = -h_n b_{n+1-j}, \qquad j = n-k+1,\ldots,n+1.$$

It should be remarked that the coefficients a_ℓ and b_ℓ are related to the coefficients of a linear multistep methods for ordinary differential equations. To see this, let scheme (1.2) be applied to an integral equation of the form $f(x) = 1 + \int_{x_0}^{x} K^*(y,f(y))dy$. Using (2.3) and (2.6) it is easily verified that (1.2) can be written as

$$\sum_{\ell=0}^{k} a_\ell f_{n+1-\ell} + h_n \sum_{\ell=0}^{k} b_\ell K^*(x_{n+1-\ell}, f_{n+1-\ell}) = 0.$$

Exactly the same formula is obtained by first writing the integral equation as the initial value problem

$$f'(x) = K^*(x,f(x)), \qquad f(x_0) = 1,$$

and applying a linear k-step method with coefficients a_ℓ and b_ℓ. Note that the choice $a_0 = -1$, $a_1 = 1$ lead to quadrature formulas having repetition factor 1 (cf. NOBLE [6]). For example the Adams-Moulton formulas generate the weights of Gregory's rule.

In the next section relation (2.5) will be converted into a system of fixed-term recurrence relations by introducing additional perturbations which are expressed in all preceding perturbations Δf_j. In section 2.2 it will be shown that a recurrence relation only containing Δf_j, $j = n+1-2k, n+2-2k,\ldots,n+1$ can be obtained provided that we restrict $K(x,y,f)$ to a special class of functions.

2.1. Introduction of additional perturbations

Let us define the additional perturbations

$$(2.7) \qquad \Delta G_n^{(i)} = \sum_{j=0}^{n} w_{n,j} R_i(x_j,f_j)\Delta f_j, \qquad n = 0,1,\ldots, \qquad i = 1,2,\ldots,r.$$

Substitution into (2.5) leads to the (r+1)(k+1)-terms relation

$$\sum_{\ell=0}^{k} [a_\ell I + b_\ell h_n P(x_{n+1-\ell}, f_{n+1-\ell})]\Delta f_{n+1-\ell} =$$

(2.5')

$$= \sum_{i=1}^{r} \sum_{\ell=0}^{k} a_\ell Q_i(x_{n+1-\ell})\Delta G^{(i)}_{n+1-\ell}.$$

In addition, we have from (2.3) for the perturbations $\Delta G_n^{(i)}$ the recurrence relations

(2.8)
$$\sum_{\ell=0}^{k} a_\ell \Delta G^{(i)}_{n+1-\ell} + h_n \sum_{\ell=0}^{k} b_\ell R_i(x_{n+1-\ell}, f_{n+1-\ell})\Delta f_{n+1-\ell} = 0$$

$i = 1,2,\ldots,r.$

By introducing the abbreviations

$$L_\ell = a_\ell I + h_n b_\ell P(x_{n+1-\ell}, f_{n+1-\ell}),$$

(2.9)
$$M_\ell^{(i)} = -a_\ell Q_i(x_{n+1-\ell}),$$

$$N_\ell^{(i)} = b_\ell h_n R_i(x_{n+1-\ell}, f_{n+1-\ell}),$$

and writing

$$\Delta G_n^{(0)} = \Delta f_n,$$

the relations (2.5') and (2.8) assume the form

$$\sum_{\ell=0}^{k} [L_\ell \Delta G^{(0)}_{n+1-\ell} + \sum_{i=1}^{r} M_\ell^{(i)} \Delta G^{(i)}_{n+1-\ell}] = 0,$$

(2.10)

$$\sum_{\ell=0}^{k} [N_\ell^{(i)} \Delta G^{(0)}_{n+1-\ell} + a_\ell \Delta G^{(i)}_{n+1-\ell}] = 0, \qquad i = 1,2,\ldots,r.$$

Let us consider the special case where L_ℓ, $M_\ell^{(i)}$ and $N_\ell^{(i)}$ do not depend on n. Then, following the theory for linear difference equations with constant coefficients we try to write the solutions of (2.10) in the form

(2.11) $$\Delta G_n^{(i)} = e^{(i)}\zeta^n, \qquad i = 0,1,\dots,r,$$

where ζ is a scalar and $e^{(i)}$ is a vector independent of n. This supposition is correct when the equations obtained by substituting (2.11) into (2.10) have a solution, i.e. when we can find a scalar ζ such that the equation

(2.10') $$\begin{pmatrix} \sum_0^k L_\ell \zeta^{k-\ell} & \sum_0^k M_\ell^{(1)}\zeta^{k-\ell} & \dots & \sum_0^k M_\ell^{(r)}\zeta^{k-\ell} \\ \sum_0^k N_\ell^{(1)}\zeta^{k-\ell} & \sum_0^k a_\ell \zeta^{k-\ell} I & & \bigcirc \\ \vdots & \bigcirc & \ddots & \\ \sum_0^k N_\ell^{(r)}\zeta^{k-\ell} & & & \sum_0^k a_\ell \zeta^{k-\ell} I \end{pmatrix} \begin{pmatrix} e^{(0)} \\ e^{(1)} \\ \vdots \\ e^{(r)} \end{pmatrix} = \vec{0}$$

has a solution $(e^{(0)},\dots,e^{(r)})$ independent of n. Representing (2.10') in the form

(2.10") $$C(\zeta)\vec{e} = \vec{0},$$

and taking for ζ a root of the equation

(2.12) $$\det[C(\zeta)] = 0,$$

it is obvious that (2.10') has a non-trivial solution $\vec{e}$. Hence, a necessary condition for (2.10) to have bounded solutions is the requirement that all solutions of (2.12) are within or on the unit circle. It can be proved that this condition is also sufficient provided that the Jordan blocks corresponding to the roots of (2.12) which are on the unit circle are 1×1-matrices [7]. In our case, however, L_ℓ, $M_\ell^{(i)}$ and $N_\ell^{(i)}$, i.e. the characteristic matrix C, depend on n and consequently the above requirement on the roots of (2.12) only has a *local* meaning. In the following the matrix C will be assumed to depend on n and denoted by C_n.

DEFINITION 2.1. For kernel functions of the class (2.2), scheme (1.2) satisfying (2.3) will be called *locally stable* at the point x_n with respect to the perturbations $\Delta G_n^{(i)}$ when all roots of (2.12) are within or on the unit circle, those on the unit circle having Jordan blocks of order 1.

It is to be expected that local stability in a sequence of points $x_n, x_{n+1}, \ldots, x_{n+m}$ implies global stability in the range $[x_n, x_{n+m}]$ provided that C_n is slowly varying in this interval with respect to the step size h_n.

In the following we will concentrate on the derivation of *local* stability criteria. These criteria often lead to conditions on the eigenvalues of the matrices $h_n K_f(x_{n+1-i}, x_{n+1-\ell}, f_{n+1-\ell})$ $i,\ell = 0,\ldots,k$. In such cases it is convenient to define the *stability region* of a method as the set of points in the space spanned by the eigenvalues $z_{n+1-i,n+1-\ell}$ of $h_n K_f(x_{n+1-i}, x_{n+1-\ell}, f_{n+1-\ell})$ where the characteristic equation (2.12) has its roots within or on the unit circle. Furthermore, it will be assumed that the matrices $Q_i(x)$ are scalar functions. We now give the following theorem:

THEOREM 2.1. *When the matrices* $Q_i(x)$ *in* (2.2) *are of the form* $q_i(x)I$, $q_i(x)$ *being a scalar function, then scheme* (1.2) *is locally stable in the sense of definition* 2.1 *if the equation*

$$\det\{\sum_{i=0}^{k} a_i \zeta^{k-i} [\sum_{\ell=0}^{k} (a_\ell I + b_\ell h_n \frac{\partial K}{\partial f}(x_{n+1-i}, x_{n+1-\ell}, f_{n+1-\ell}))\zeta^{k-\ell}]\} = 0 \tag{2.12}$$

has its roots within the unit circle or on the unit circle, those on the unit circle having Jordan blocks of order 1.

PROOF. By elementary manipulations with the rows in the matrix C_n we have

$$\det[C_n(\zeta)] = \det\{(\sum_{i=0}^{k} a_i I \zeta^{k-i})^{r-1}\} * \det\{\sum_{i=0}^{k} a_i \zeta^{k-i} *$$

$$[\sum_{\ell=0}^{k} (a_\ell I + b_\ell h_n \frac{\partial K}{\partial f}(x_{n+1-i}, x_{n+1-\ell}, f_{n+1-\ell}))\zeta^{k-\ell}]\} = 0.$$

Let $\tilde{\zeta}$ be a root of $\Sigma_{i=0}^{k} a_i \zeta^{k-i}$ with $|\tilde{\zeta}| = 1$ (for instance, $\tilde{\zeta} = 1$ is always a root by virtue of (2.3)). If the matrices in (2.10') have dimension s then $\tilde{\zeta}$ is a root of multiplicity (r-1)s. Solving (2.10") with $\zeta = \tilde{\zeta}$ yields:

$$\vec{e}_0^{(0)} = \vec{0}, \quad \sum_{\ell=0}^{k} M_\ell^{(1)} \tilde{\zeta}^{k-\ell} \vec{e}^{(1)} + \ldots + \sum_{\ell=0}^{k} M_\ell^{(r)} \tilde{\zeta}^{k-\ell} \vec{e}^{(r)} = \vec{0}.$$

Hence, we can find (r-1)s independent vectors satisfying (2.10"), or in other words, the Jordan blocks corresponding to $\tilde{\zeta}$ are 1×1-matrices. □

The problem now is how to derive practical criteria from theorem 2.1. When we proceed as in the stability analysis of integration methods for ordinary differential equations, the Jacobian matrices $K_f(x_{n+1-i}, x_{n+1-\ell}, f_{n+1-\ell})$, in the characteristic equation (2.12) are replaced by a locally constant matrix J (the "slowly varying Jacobian" approach), which leads to the characteristic equation

$$\sum_{\ell=0}^{k} a_\ell \zeta^{k-\ell} \sum_{\ell=0}^{k} (a_\ell + b_\ell z) \zeta^{k-\ell} = 0, \tag{2.13}$$

where z denotes an eigenvalue of $h_n J$. Since this equation is just the characteristic equation of the linear multistep method with coefficients a_ℓ and b_ℓ when it is applied to the differential equation df/dx = Jf, stability criteria can directly be derived from the stability theory for ordinary differential equations. This immediately suggests to choose scheme (1.2) such that it corresponds to a multistep method with good stability properties. Such methods are the Curtiss-Hirschfelder or backward differentiation formulas [2] defined by

$$a_0 = -1, \quad \sum_{\ell=1}^{k} (1-\ell)^j a_\ell + j b_0 = 1, \qquad j = 0,1,\ldots,k,$$

$$b_\ell = 0, \qquad \ell = 1,2,\ldots,k.$$

The stability regions of these formulas contain the whole left half plane for k ≤ 2 and almost the whole left half plane (except for a small region near the imaginary axis) for k = 3,4,5 and 6. In order to make use of these excellent stability properties one should find the corresponding weights $w_{n,j}$ by solving the relations (2.3) and (2.6). In [8] solutions are given and the resulting quadrature formulas are investigated.

We should bear in mind, however, that the replacement of $K_f(x_{n+1-i}, x_{n+1-\ell}, f_{n+1-\ell})$ by a constant matrix J is dubious when the Jacobian matrix is a rapidly changing function of x, y and f. In order to obtain more rigorous stability criteria additional information about the integral equation should be provided.

In section 2.3, stability results will be derived in terms of the eigenvalues of the Jacobian matrices $K_f(x_{n+1-i}, x_{n+1-\ell}, f_{n+1-\ell})$ for two specific classes of kernel functions and a specific integration scheme. Firstly, however, we describe in the next section an approach to generate recurrence relations with a fixed number of terms only containing perturbations Δf_j.

2.2. Recurrence relations without additional perturbations

When the Jacobian matrix of the kernel function has the form

$$\frac{\partial K}{\partial f}(x,y,f) = P(y,f) + xR(y,f) \tag{2.14}$$

it is possible to obtain a recurrence relation only containing a fixed number of perturbations Δf_j. This will be shown as follows. For kernel functions satisfying (2.14) relation (2.4) can be written in the form

$$\begin{aligned} \Delta f_{n+1-\ell} &= \sum_{j=0}^{n+1} w_{n+1-\ell,j}\{P(x_j,f_j) + x_{n+1}R(x_j,f_j)\}\Delta f_j \\ &\quad - \ell h \sum_{j=0}^{n+1} w_{n+1-\ell,j} R(x_j,f_j)\Delta f_j, \qquad n+1-\ell \geq k,\ 0 \leq \ell \leq k, \end{aligned} \tag{2.15}$$

where we have assumed $h_n = h$, i.e. a constant step size. Taking suitable linear combinations of (2.15) and using (2.3) and (2.6) we obtain the relations

$$\sum_{\ell=0}^{k} a_\ell \Delta f_{n+1-\ell} +$$

(2.16) $$+ h \sum_{\ell=0}^{k} b_\ell \{P(x_{n+1-\ell}, f_{n+1-\ell}) + x_{n+1} R(x_{n+1-\ell}, f_{n+1-\ell})\}\Delta f_{n+1-\ell}$$

$$+ h \sum_{\ell=0}^{k} \ell a_\ell \sum_{j=0}^{n+1} w_{n+1-\ell,j} R(x_j, f_j)\Delta f_j = 0, \qquad n \geq 2k-1.$$

Since (2.16) holds for all $n \geq 2k-1$ we may write down $k+1$ consecutive relations as follows

$$\sum_{\ell=0}^{k} a_\ell \Delta f_{n+1-i-\ell} + h \sum_{\ell=0}^{k} b_\ell \{P(x_{n+1-i-\ell}, f_{n+1-i-\ell}) +$$

(2.17) $$+ x_{n+1-i} R(x_{n+1-i-\ell}, f_{n+1-i-\ell})\}\Delta f_{n+1-i-\ell} +$$

$$+ h \sum_{\ell=0}^{k} \ell a_\ell \sum_{j=0}^{n+1-i} w_{n+1-i-\ell,j} R(x_j, f_j)\Delta f_j = 0,$$

$$n+1-i \geq 2k, \quad 0 \leq i \leq k.$$

Again taking linear combinations of (2.17) yields the relations

$$\sum_{i=0}^{k} a_i \sum_{\ell=0}^{k} [a_\ell + hb_\ell \{P(x_{n+1-i-\ell}, f_{n+1-i-\ell}) +$$

(2.18) $$+ x_{n+1-i} R(x_{n+1-i-\ell}, f_{n+1-i-\ell})\}]\Delta f_{n+1-i-\ell} +$$

$$+ h \sum_{i=0}^{k} a_i \sum_{\ell=0}^{k} \ell a_\ell \sum_{j=0}^{n+1-i} w_{n+1-i-\ell,j} R(x_j, f_j)\Delta f_j = 0,$$

$$n \geq 3k-1.$$

By interchanging the summations of the last term of the left-hand side of (2.18) this term reduces to

$$h \sum_{j=0}^{n+1} \sum_{\ell=0}^{k} \ell a_\ell \Big(\sum_{i=0}^{k} a_i w_{n+1-i-\ell,j} \Big) R(x_j,f_j)\Delta f_j =$$

$$h \sum_{\ell=0}^{k} \ell a_\ell \sum_{j=0}^{n+1-\ell} \Big(\sum_{i=0}^{k} a_i w_{n+1-i-\ell,j} \Big) R(x_j,f_j)\Delta f_j =$$

$$- h^2 \sum_{\ell=0}^{k} \ell a_\ell \sum_{i=0}^{k} b_i R(x_{n+1-i-\ell}, f_{n+1-i-\ell})\Delta f_{n+1-i-\ell},$$

where we have used (2.3) and (2.6). Substitution into (2.18) yields the recurrence relation

$$\sum_{i=0}^{k} a_i \sum_{\ell=0}^{k} [a_\ell + hb_\ell\{P(x_{n+1-i-\ell}, f_{n+1-i-\ell}) +$$

(2.19) $$+ (x_{n+1-i} - ih)R(x_{n+1-i-\ell}, f_{n+1-i-\ell})\}]\Delta f_{n+1-i-\ell} = 0,$$

$$n \geq 3k-1.$$

Note that Δf_{n+1} only depends on Δf_j for $n+1-2k \leq j \leq n$. The definition of local stability, given in section 2.1, should be modified in an obvious manner, viz. the kernel functions are restricted to the class (2.14) and the space of perturbations only contains Δf_j. From (2.19) we finally have the following theorem.

THEOREM 2.2. *Scheme* (1.2) *satisfying* (2.3) *is locally stable in the sense of definition* (2.1) *(with obvious modifications) if the equation*

(2.20) $$\det\Big\{ \sum_{i=0}^{k} a_i \zeta^{k-i} * \Big[\sum_{\ell=0}^{k} \Big(a_\ell I + b_\ell h \frac{\partial K}{\partial f}(x_{n+1-2i}, x_{n+1-i-\ell}, f_{n+1-i-\ell})\Big)\zeta^{k-\ell} \Big]\Big\} = 0$$

has its roots within the unit circle or on the unit circle, those on the unit circle having Jordan blocks of order 1.

Comparing (2.20) and (2.12') we see that in (2.20) more of the "history" of

$\frac{\partial K}{\partial f}$ is taken into account. However, when the Jacobian matrix $\frac{\partial K}{\partial f}$ is locally approximated by a constant matrix J it is easily seen that (2.20) is identical to (2.13).

We remark that the analysis presented in this section can be generalized for the class of kernel functions satisfying

$$\frac{\partial K}{\partial f}(x,y,f) = P(y,f) + \sum_{i=1}^{r} x^i R_i(y,f),$$

by repeatly taking suitable linear combinations. The resulting recurrence relation for Δf_{n+1} contains $k(r+1) + 1$ terms Δf_j. However, the resulting stability conditions are difficult to derive even for the case $k = 1$.

2.3. Derivation of stability conditions

In the derivation of stability conditions from the theorems 2.1 and 2.2 the following lemma is frequently used:

LEMMA 2.1. *Let* $F(\zeta)$ *be a matrix-valued function of the scalar* ζ *with eigenvalues* $\phi_j(\zeta)$, $j = 1,2,\ldots,s$. *Then the roots of the equation*

$$\det[F(\zeta)] = 0$$

are within or on the unit circle when the roots of the equations

$$\phi_j(\zeta) = 0, \qquad j = 1,2,\ldots,s$$

are within or on the unit circle.

PROOF. Let $\tilde{\zeta}$ be a root of $\det[F(\zeta)] = 0$ then the matrix $F(\tilde{\zeta})$ necessarily has a zero eigenvalue. Since the eigenvalues of $F(\tilde{\zeta})$ are of the form $\phi_j(\tilde{\zeta})$ we have $\phi_j(\tilde{\zeta}) = 0$ for some j. Thus, be requiring that all roots ζ of $\phi_j(\zeta) = 0$ satisfy $|\zeta| \leq 1$ for all j, the roots of $\det[F(\zeta)] = 0$ certainly are within or on the unit circle. □

2.3.1. Jacobian matrices with locally constant eigensystems

When the eigensystems of the matrices $K_f(x_{n+1-i}, x_{n+1-\ell}, f_{n+1-\ell})$ coincide for $i,\ell = 0,1,\dots,k$ (e.g. in case of *scalar* integral equations), it is immediate from theorem 2.1 and lemma 2.1 that we have local stability if the roots ζ of the equation

$$(2.21) \qquad \sum_{i,\ell=0}^{k} a_i [a_\ell + b_\ell z_{n+1-i,n+1-\ell}] \zeta^{2k-i-\ell} = 0,$$

satisfy $|\zeta| \leq 1$ for all eigenvalues $z_{n+1-i,n+1-\ell}$ of $h_n K_f(x_{n+1-i}, x_{n+1-\ell}, f_{n+1-\ell})$.

For small values of k this equation easily gives the stability region in the eigenvalue space $\{z_{n+1-i,n+1-\ell}\}_{i,\ell=0}^{k}$. We shall illustrate this by analyzing the trapezoidal rule defined by

$$(2.2) \qquad w_{n,0} = 2w_{n,1} = 2w_{n,2} = \dots = 2w_{n,n-1} = w_{n,n} = \frac{1}{2} h.$$

This quadrature rule satisfies (2.3) with $k = 1$, $a_0 = -a_1 = -1$. From this it follows that $b_0 = b_1 = \frac{1}{2}$, so that equation (2.21) assumes the form

$$(2.23) \qquad (1 - \frac{1}{2} z_{n+1,n+1})\zeta^2 - (2 + \frac{1}{2} z_{n+1,n} - \frac{1}{2} z_{n,n+1})\zeta + (1 + \frac{1}{2} z_{n,n}) = 0.$$

Let the eigenvalues z be real then by the Hurwitz criterion we arrive at the stability region

$$(2.24) \qquad \begin{aligned} &z_{n+1,n+1} \leq 0, \\ &z_{n+1,n+1} + z_{n,n} \leq 0, \\ &z_{n+1,n+1} - z_{n,n} + z_{n+1,n} - z_{n,n+1} \leq 0, \\ &z_{n+1,n+1} - z_{n,n} + z_{n,n+1} - z_{n+1,n} \leq 8, \end{aligned}$$

Next we consider the stability conditions resulting from the analysis presented in section 2.2. From theorem 2.2 and lemma 2.1 it follows that we have local stability if the roots of the equation

$$(2.25) \qquad \sum_{i,\ell=0}^{k} a_i[a_\ell + b_\ell z_{n+1-2i,n+1-i-\ell}]\zeta^{2k-i-\ell} = 0,$$

satisfy $|\zeta| \leq 1$ for all eigenvalues $z_{n+1-2i,n+1-i-\ell}$. In the case of the trapezoidal rule we arrive at the stability region

$$(2.26) \qquad \begin{aligned} & z_{n+1,n+1} \leq 0, \\ & z_{n+1,n+1} + z_{n-1,n-1} \leq 0, \\ & z_{n+1,n+1} - z_{n-1,n-1} + z_{n+1,n} - z_{n-1,n} \leq 0, \\ & z_{n+1,n+1} - z_{n-1,n-1} - z_{n+1,n} + z_{n-1,n} \leq 8. \end{aligned}$$

It is not surprising that the regions defined by (2.24) and (2.26) are different because the spaces of perturbations, appearing in the definition of local stability, are different.

The preceding derivations become increasingly difficult for larger values of k. In such cases one may get a rough impression of the stability region by applying the "slowly varying Jacobian" approach mentioned above. In case of the trapezoidal rule this would give the familiar condition (for complex eigenvalues)

$$(2.24') \qquad \mathrm{Re}\, z_{n+1,n+1} \leq 0.$$

Although the conditions resulting from (2.21) and (2.25) depend on n, i.e. only have a local meaning, there is an important class of kernel functions for which the analysis of section 2.1 and 2.2 may give conditions for global stability. This is investigated in the next section.

2.3.2. Convolution kernels

An important class of integral equations has kernels of the form

$$(2.27) \qquad K(x,y,f) = k^*(x-y)Af,$$

where k^* is a polynomial in (x-y) and A is a matrix with constant elements. Evidently the Jacobian of (2.27) can be presented in the form (2.2) with scalar functions $Q_i(x)$. Furthermore, the eigensystem of $\partial K/\partial f$ does not depend on x, y and f so that we arrive at the characteristic equation (2.21) with

$$z_{n+1-i,n+1-\ell} = h_n k^*(x_{n+1-i} - x_{n+1-\ell})\alpha$$

where α runs through the eigenvalues of A. In the case where constant integration steps h are used equation (2.21) reduces to

$$(2.28) \qquad \sum_{i,\ell=0}^{k} a_i[a_\ell + b_\ell hk^*((\ell-i)h)\alpha]\zeta^{2k-i-\ell} = 0.$$

Note that the requirement that (2.28) has its roots within the unit circle not only implies *local* stability but also *global* stability as the characteristic matrix C does not depend on n. Secondly, we note that the analysis of section 2.2 can be applied only when $k^*(x-y) = \gamma_1 + \gamma_2(x-y)$. Therefore, in order to compare the results of section 2.1 and 2.2, we will derive stability conditions when the scheme is applied to such kernel functions. It is easily verified that for such kernel functions (2.20) is identical to (2.28) and has the form

$$(2.28') \qquad \sum_{i=0}^{k} a_i\zeta^{k-i} \sum_{\ell=0}^{k} [a_\ell + b_\ell h\alpha\gamma_1]\zeta^{k-\ell} + h^2\alpha\gamma_2 \sum_{i,\ell=0}^{k} a_i b_\ell (\ell-i)\zeta^{2k-i-\ell} = 0.$$

The second double sum in this equation is due to taking into account the variation of the Jacobian matrix with x and y. Especially, when γ_1 is small, i.e. $\gamma_1 = \mathcal{O}(h)$, these terms cannot be neglected so that equation (2.28') may differ considerably from the "multistep equation" (2.13).

Finally, we give the conditions for global stability of the trapezoidal rule in the case of a convolution kernel (compare (2.26)).

$$(2.26') \qquad h\alpha\gamma_1 \le 0, \quad h^2\alpha\gamma_2 \le 0, \quad h^2\alpha\gamma_2 \ge -4.$$

From the last condition we conclude that the stepsize h is *restricted* by $h^2 \le \frac{4}{|\alpha\gamma_2|}$, α being an eigenvalue of A. Numerical experiments, reported in the next section, confirm this result.

3. NUMERICAL ILLUSTRATION

In this section we will verify the stability analysis when applied to the trapezoidal rule. In particular we are interested in the difference between conditions (2.24) and (2.26) resulting from a different analysis. It should be noted however that (2.26) is applicable only if $K_{xxf} = 0$.

In order to illustrate the analysis we specify a number of scalar integral equations. For each problem we will check the conditions for stability yielding a prediction for stable or unstable computation. A final numerical experiment will verify this prediction. All problems were solved using a constant stepsize h, the range of integration was 100h.

Problem 1:

$$K(x,y,f) = (-a + bx + cy)f;$$

$$g(x) = (1-c)\sin x + a - bx + ((b+c)x - a)\cos x$$

$$\text{solution } f(x) = \sin x.$$

For this set of problems the (local) stability conditions (2.24) and (2.26) are

$$(3.1) \qquad 100\,h(b+c) \le a; \quad bh^2 \le 0; \quad ch^2 \le 4,$$

and

$$(3.2) \qquad 100\,h(b+c) \le a; \quad 2bh^2 + ch^2 \le 0; \quad ch^2 \le 4.$$

Note that (3.1) is more stringent that (3.1) as it allows the parameter b only to be negative. We have chosen the following values of the parameters:

1a) $a = 1001, \quad b = -900, \quad c = 1000, \quad h = 0.1$

The conditions (3.1) and (3.2) predict instability. The numerical solution was indeed unstable. The true error was amplified by a factor of approximately -4.

1b) $a = 1501, \quad b = -50, \quad c = 200, \quad h = 0.1.$

Condition (3.1) predicts stability, whereas (3.2) predicts an unstable behaviour. No severe instabilities were developed during the computation, only a small increase in the absolute error was detected. Hence, the numerical solution did not give a decisive answer to the question whether it was stable or not. In fact, it was difficult to find an example for which the numerical solution was strongly unstable.

1c) $a = 1, \quad b = -400, \quad c = 400, \quad h = 0.1.$

Conditions (3.1) and (3.2) reveal that for this choice of the parameters we have a point on the boundary of the stability region. Indeed, inspection of the true error confirmed that the amplification factor was approximately -1.

1d) convolution kernel.

$$a = 1/4, \quad b = -320, \quad c = 320, \quad h = 1/8.$$

For convolution kernels conditions (3.1) and (3.2) coincide and predict global instability. In every step, the absolute error was amplified by a factor $\simeq -2.8$, indicating a severe instability.

1e) convolution kernel

$$a = 1/4, \quad b = -128, \quad c = 128, \quad h = 1/8.$$

Both conditions predict stability. The numerical results confirmed this behaviour.

Problem 2. (nonlinear)

$$K(x,y,f) = -x^2 f^3; \quad g(x) = x^2 + \frac{1}{7} x^9$$

$$\text{solution: } f(x) = x^2, \quad h = 0.1.$$

It is easily seen that (2.26) gives no decisive answer, since it is not applicable, (2.24) predicts stability. No instabilities were developed during the computation.

From these experiments we conclude that the stability conditions (2.24) and (2.26) give a good prediction for the global numerical behaviour, although they result from a local analysis.

4. CONCLUDING REMARKS

An important feature of this paper is the considerable extension of the class of kernel functions for which a stability analysis can be carried out. However, the derivation of stability conditions from our analysis is elaborate even in the case of scalar equations. Therefore, we recommend to apply the present analysis only to schemes which have good stability properties with respect to the test equation $f(x) = 1 + \int_0^x f(y)dy$. Since applying our analysis to more general kernel functions may lead to more rigorous stability conditions (compare the trapezoidal rule), it must be regarded as a necessary second sieve in order to select an appropriate scheme for the solution of Volterra integral equations.

The class of quadrature rules which we have considered is related to the class of linear multistep methods for solving ordinary differential equations and, therefore, the quadrature weights do have a certain structure, although this is not necessarily a line by line repetition. On the other hand, not all composite rules, investigated by BAKER & KEECH [1], in particular those having repetition factor unity, fall within this class. However, a generalization of our analysis so as to treat this class of schemes as well, is subject to further research.

ACKNOWLEDGEMENT

The authors are grateful to H.J.J. te Riele for carefully reading the manuscript and for his useful comments.

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